Problem 1.64

In case you're not persuaded that $\nabla^2(1/r) = -4\pi\delta^3(\mathbf{r})$ (Eq. 1.102, with $\mathbf{r}' = \mathbf{0}$ for simplicity), try replacing r by $\sqrt{r^2+\epsilon^2}$, and watching what happens as $\epsilon \to 0$.[16] Specifically, let

$$D(r,\epsilon) \equiv -\frac{1}{4\pi}\nabla^2 \frac{1}{\sqrt{r^2+\epsilon^2}}.$$

To demonstrate that this goes to $\delta^3(\mathbf{r})$ as $\epsilon \to 0$:

(a) show that $D(r,\epsilon) = (3\epsilon^2/4\pi)(r^2+\epsilon^2)^{-5/2}$;

(b) check that $D(0,\epsilon) \to \infty$, as $\epsilon \to 0$;

(c) check that $D(r,\epsilon) \to 0$, as $\epsilon \to 0$, for all $r \neq 0$;

(d) check that the integral of $D(r,\epsilon)$ over all space is 1.

Solution

By definition, the delta function $\delta^3(\mathbf{r})$ satisfies the following two properties.

$$(1) \quad \delta^3(\mathbf{r}) = \begin{cases} 0 & \text{if } |\mathbf{r}| \neq 0 \\ \infty & \text{if } |\mathbf{r}| = 0 \end{cases}$$

$$(2) \quad \int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \delta^3(\mathbf{r})\, d\mathbf{r} = 1$$

The aim in this problem is to show that $D(r,\epsilon)$ satisfies these two properties as $\epsilon \to 0$. Start by simplifying $D(r,\epsilon)$.

$$\begin{aligned}
D(r,\epsilon) &= -\frac{1}{4\pi}\nabla^2 \frac{1}{\sqrt{r^2+\epsilon^2}} \\
&= -\frac{1}{4\pi}\left\{\frac{1}{r^2}\frac{\partial}{\partial r}\left[r^2\frac{\partial}{\partial r}\left(\frac{1}{\sqrt{r^2+\epsilon^2}}\right)\right] + \overbrace{\frac{1}{r^2\sin\theta}\frac{\partial}{\partial\theta}\left[\sin\theta\frac{\partial}{\partial\theta}\left(\frac{1}{\sqrt{r^2+\epsilon^2}}\right)\right]}^{=0}\right. \\
&\qquad\qquad \left. + \frac{1}{r^2\sin^2\theta}\underbrace{\frac{\partial^2}{\partial\phi^2}\left(\frac{1}{\sqrt{r^2+\epsilon^2}}\right)}_{=0}\right\} \\
&= -\frac{1}{4\pi r^2}\frac{d}{dr}\left\{r^2\left[-\frac{r}{(r^2+\epsilon^2)^{3/2}}\right]\right\} \\
&= -\frac{1}{4\pi r^2}\frac{d}{dr}\left[-\frac{r^3}{(r^2+\epsilon^2)^{3/2}}\right] \\
&= -\frac{1}{4\pi r^2}\left[-\frac{3\epsilon^2 r^2}{(r^2+\epsilon^2)^{5/2}}\right] \\
&= \frac{3\epsilon^2}{4\pi(r^2+\epsilon^2)^{5/2}}
\end{aligned}$$

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[16] This problem was suggested by Frederick Strauch.

Check that $D(r,\epsilon)$ satisfies property (1) in the limit as $\epsilon \to 0$.

$$r = 0: \quad \lim_{\epsilon\to 0} D(0,\epsilon) = \lim_{\epsilon\to 0} \frac{3\epsilon^2}{4\pi(\epsilon^2)^{5/2}} = \lim_{\epsilon\to 0} \frac{3\epsilon^2}{4\pi\epsilon^5} = \lim_{\epsilon\to 0} \frac{3}{4\pi\epsilon^3} = \infty$$

$$r \neq 0: \quad \lim_{\epsilon\to 0} D(r,\epsilon) = \lim_{\epsilon\to 0} \frac{3\epsilon^2}{4\pi(r^2+\epsilon^2)^{5/2}} = \frac{3(0)^2}{4\pi(r^2)^{5/2}} = \frac{0}{4\pi r^5} = 0$$

Check that $D(r,\epsilon)$ satisfies property (2).

$$\iiint\limits_{\text{all space}} D(r,\epsilon)\,dV = \int_0^\pi \int_0^{2\pi} \int_0^\infty \frac{3\epsilon^2}{4\pi(r^2+\epsilon^2)^{5/2}} (r^2 \sin\theta\, dr\, d\phi\, d\theta)$$

$$= \frac{3\epsilon^2}{4\pi} \left(\int_0^\pi \sin\theta\, d\theta\right) \left(\int_0^{2\pi} d\phi\right) \left[\int_0^\infty \frac{r^2\, dr}{(r^2+\epsilon^2)^{5/2}}\right]$$

Make a trigonometric substitution.

$$r = \epsilon \tan\theta \quad \to \quad r^2 + \epsilon^2 = \epsilon^2(1+\tan^2\theta) = \epsilon^2 \sec^2\theta$$

$$dr = \epsilon \sec^2\theta\, d\theta$$

As a result,

$$\iiint\limits_{\text{all space}} D(r,\epsilon)\,dV = \frac{3\epsilon^2}{4\pi}(2)(2\pi) \int_0^{\pi/2} \frac{(\epsilon\tan\theta)^2(\epsilon\sec^2\theta\, d\theta)}{(\epsilon^2\sec^2\theta)^{5/2}}$$

$$= 3\epsilon^2 \int_0^{\pi/2} \frac{(\epsilon^2\tan^2\theta)(\epsilon\sec^2\theta\, d\theta)}{\epsilon^5\sec^5\theta}$$

$$= 3\int_0^{\pi/2} \tan^2\theta \cos^3\theta\, d\theta$$

$$= 3\int_0^{\pi/2} \sin^2\theta \cos\theta\, d\theta$$

$$= 3\int_0^{\pi/2} \sin^2\theta\, d(\sin\theta)$$

$$= 3\frac{\sin^3\theta}{3}\bigg|_0^{\pi/2}$$

$$= \sin^3\frac{\pi}{2} - \sin^3 0$$

$$= 1 - 0$$

$$= 1.$$